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Multiparameter R-matrix for orthosymplectic groups

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Abstract. We consider the multiparameter deformation of a quantum mechanical phase space associated with n bosonic and m fermionic coordinates. This generates $(n+m)\times(n+m-1)/2+1$ parameter solution of the quantum Yang-Baxter equation for $OSp(2m/2n)$. The multiparameter R-matrices for $O(2m)$ and $Sp(2n)$ found recently by Schirrmacher are obtained as special cases. As an example we consider $OSp(2/2)$.

Quantum group theory, which has found applications in many areas of physics and mathematics, is based on a distinguished class of Hopf algebras associated with spectral parameter independent limits of R-matrix solutions of quantum Yang-Baxter equations (QYBE) for Lie groups and supergroups. In this work we shall deduce a multiparameter R-matrix solution for $OSp(2m/2n)$, where the total number of deformation parameters, $(n+m)(n+m-1)/2+1$, agrees with (super-analogue of) the general observation of Reshitikhin [1] relating to multiparameter quantization of groups. Our solution includes as a special case, the multiparameter R-matrix solutions of QYBE for C and D series recently given by Schirrmacher [2].

Zumino [3] considers one-parameter deformation of the quantum mechanical phase space with bosonic or fermionic variables. We shall first show how, by tracing a development parallel to Zumino's, we can be led to the same multiparameter R-matrix for the C series which Schirrmacher has deduced by explicit integration of the underlying Yang-Baxter algebra.

Following Zumino let us consider an n-dimensional quantum hyperplane defined in terms of coordinates z^i $(i=1,2,\ldots,n)$. Its associated quantum mechanical phase space is given in terms of variables y^a $(a=1,2,\ldots,2n)$; these secondary variables y^a are derived from the primary variables z^i: $y^i=p_{\hat{i}}$, $y^{n+i}=z^i$ $(i=1,2,\ldots,n)$. Here $\hat{i}=n+1-i$ and p_i denotes a momentum variable which is canonically conjugate to z^i. It is clear that the phase space coordinate $y^{i'}$ is canonically conjugate to y^i, where $i'+i=2n+1$. The momenta p_i are, up to multiplicative factors, nothing but derivatives of the quantum hyperplane $\partial_i=\partial/\partial z^i$. Table 1 gives all the rules [4] for a consistent multiparameter deformation of the classical algebra of coordinates and derivatives. With the help of this multiparameter differential calculus of Schirrmacher (Wess and Zumino [5] had considered, as an example of their general considerations, one particular family of one-parameter q-deformation which was obtained by equating all the deformation parameters), one can easily derive all the commutation relations between the phase space variables y^a. We may distinguish the two cases:

(i) y^a and y^b are a pair of non-conjugate variables, by definition $a+b\neq 2n+1$. Then

$$y^a y^b = f_{ab} y^b y^a \tag{1}$$

Table 1. Multiparameter commutation relations defining covariant differential calculus on the quantum hyperplane. The lack of commutativity of plane coordinates z^i is parametrized in terms of $n(n-1)/2$ (possibly complex) deformation parameters denoted q_{ij}, and r denotes the Lie algebra deformation parameter. The coordinate indices i and j may be chosen arbitrarily such that $i<j$. The summation convention is understood. By definition $\Theta_i^k=0$ if $k\leqslant i$ and 1 otherwise.

$$z^iz^j=q_{ij}z^jz^i$$
$$\partial_iz^j=(r^2/q_{ij})z^j\partial_i$$
$$\partial_jz^i=q_{ij}z^i\partial_j$$
$$\partial_iz^i=1+r^2z^i\partial_i+(r^2-1)\Theta_i^kz^k\partial_k$$
$$\partial_i\partial_j=(q_{ij}/r^2)\partial_j\partial_i.$$

with the restriction that $a<b$. The coefficients f_{ab} satisfy

$$f_{ba}=\delta_{ab}+(r^2/f_{ab})\Theta_{ba}+f_{ba}\Theta_{ab}$$
$$f_{ab}=f_{a'b'}=\frac{r^2}{f_{ab'}}=\frac{r^2}{f_{a'b}}. \tag{2}$$

Clearly, only $n(n-1)/2$ fs comprise independent deformation parameters. One finds for instance that

$$f_{ab}=r^2/q_{ab} \qquad 1\leqslant a<b\leqslant n.$$

(ii) y^a and y^b are a pair of conjugate variables, by definition $a'=b$. Let $1\leqslant a\leqslant n$, then (summation over b is understood)

$$r^2y^{a'}y^a=c_a+y^ay^{a'}-(r^2-1)r^{\hat{a}-\hat{b}}y^by^{b'}\Theta^a{}_b. \tag{3}$$

Besides the homogeneous quadratic terms we also have the inhomogeneous constant terms $c_a=\mathrm{i}\hbar r^{2-a}$ in (3). An important property of this algebra is that (summation over a through 1 to $2n$ is understood)

$$\varepsilon_{a'}r^{\hat{a}}y^{a'}y^a=-r^{n-2}(c_1+c_2/r+\ldots+c_n/r^{n-1})=-\mathrm{i}\hbar[n]. \tag{4}$$

Here $[n]=(r^n-r^{-n})/(r-r^{-1})$, and by definition

$$\hat{a}=n+1-a \qquad \hat{a}'=-\hat{a} \qquad \varepsilon_a:=1 \qquad \varepsilon_{a'}:=-1.$$

Equation (4) is the symplectic constraint on the associative Hopf algebra generated by y^a.

If we ignore the $\mathrm{O}(\hbar)$ constant terms, the aforementioned quadratic algebra is succinctly given using the R-matrix, i.e. $y\otimes y=r^{-1}PRy\otimes y$. In a matrix form the quadratic algebra is explicitly given by

$$y^by^a=\frac{1}{r}R^{ab}{}_{cd}y^cy^d \tag{5}$$

where $R^{ab}{}_{cd}$ is precisely the $(2n)^2\times(2n)^2$ R-matrix of Schirrmacher:

$$R^{ab}{}_{cd}=\frac{r}{f_{ab}}\delta^a{}_c\delta^b{}_d+(r-r^{-1})[\delta^a{}_d\delta^b{}_c\Theta^{ab}+\delta^{ab'}\delta_{cd'}\Theta^a{}_c\mathscr{A}^a{}_c] \tag{6}$$

with

$$\mathscr{A}^a{}_b=\varepsilon_{a'}\varepsilon_br^{\hat{a}-\hat{b}}.$$

We have given multiparameter extension of the consideration analogous to Zumino's. It is not hard to generalize these phase space considerations to superspace. We shall do this by taking into consideration graduation of the underlying vector space. As an obvious generalization of the z^i coordinate space consider an $(n+m)$-dimensional quantum vector superspace defined in terms of supernumerary variables z^I $(I=1,2,\ldots,n+m)$. The Grassmann degree $\pi(I)$ of bosonic coordinates $z^1,z^2,\ldots,z^n$ is zero; and that of fermionic coordinates z^{n+1}, $z^{n+2},\ldots,z^{n+m}$ is one. This implies in turn that squares of fermionic coordinates must vanish. The corresponding quantum mechanical phase space with n bosonic and m fermionic degrees of freedom may be defined in terms of the following $2n+2m$ variables x^A:

$$\begin{aligned} x^\alpha &= \hbar r^{-(\alpha-1)}\partial_{n+m+1-\alpha} && 1\leqslant\alpha\leqslant m \\ x^{m+i} &= -\mathrm{i}\hbar r^{i-m}\partial_{n+1-i} && 1\leqslant i\leqslant n \\ x^{m+n+I} &= z^I && 1\leqslant I\leqslant n+m. \end{aligned} \tag{7}$$

The derivatives are multiplied by suitable powers of r in order to ensure reality of $x^1,x^2,\ldots,x^{n+m}$. Given the relation of xs to zs, the appropriate generalization of the foregoing considerations can be worked out with the help of all the rules [6] collected in table 2.

Table 2. Generalization of the rules given in table 1 to superspace. The Grassmann degrees of $m(n)$ indices are 1(0). The supernumerary indices are denoted by capital letters $I,J,\ldots$, etc, such that $I<J$. The sign(I,J) stands for $(-1)^{\pi(I)\pi(J)}$ and sign(I,I) for $(-1)^{\pi(I)}$.

$$z^Iz^J=\mathrm{sign}(I,J)q_{IJ}z^Jz^I$$
$$(z^{n+1})^2=(z^{n+2})^2=\ldots=(z^{n+m})^2=0$$
$$\partial_Iz^J=\mathrm{sign}(I,J)\frac{r^2}{q_{IJ}}z^J\partial_I$$
$$\partial_Jz^I=\mathrm{sign}(I,J)q_{IJ}z^I\partial_J$$
$$\partial_Iz^I=1+\frac{\mathrm{sign}(I,I)r^2}{(r^2)^{\pi(I)}}z^I\partial_I+(r^2-1)\Theta_I^Kz^K\partial_K$$
$$\partial_I\partial_J=\mathrm{sign}(I,J)\frac{q_{IJ}}{r^2}\partial_J\partial_I.$$

For the (anti-)commutation relations involving non-conjugate supercoordinates x^A and x^B, it can be checked that

$$x^Ax^B=\mathrm{sign}(A,B)f_{AB}x^Bx^A \tag{8}$$

if A is less than or equal to B. Here the signature is ±1 depending on whether $\pi(A)\pi(B)$ is zero or 1. The coefficients f_{AB} which depend on $(n+m)(n+m-1)/2+1$ deformation parameters are required to satisfy

$$\begin{aligned} f_{BA} &= (r^2)^{\pi(A)}\delta_{AB}+\Theta_{BA}r^2/f_{AB}+f_{BA}\Theta_{AB} \\ f_{AB} &= f_{A'B'}=\frac{\gamma^2}{f_{A'B}}=\frac{\gamma^2}{f_{AB'}} \\ f_{ab} &= \frac{r^2}{q_{ab}} \qquad 1\leqslant a<b\leqslant n+m. \end{aligned} \tag{9}$$

On the other hand, for a pair of conjugate supercoordinates x^A and $x^{A'}$ the (anti-)commutation relations can be written in the following manner (summation over the repeated index c is implied and $1 \leq A \leq n+m$):

$$x^{A'}x^A = \tilde{c}_A + \frac{(-1)^{\pi(A)}}{f_{AA'}} x^A x^{A'} + (1-r^{-2}) x^{A'} x^A \Theta^{AA'} + (1-r^{-2})(-1)^{\pi(A)} i^{(\pi(A)-\pi(C))} \mathscr{A}^A{}_C \Theta^A{}_C x^C x^{C'} \tag{10}$$

where $\mathscr{A}^A{}_B = \varepsilon_{A'}\varepsilon_B r^{\hat{A}-\hat{B}}$.

The 'hatted' supernumerary indices are given by $\hat{a} = -\hat{a}'$, together with

$$\hat{1}, \hat{2}, \ldots, \hat{m}, \widehat{m+1}, \ldots, \widehat{m+n} = n-(m-1), n-(m-2), \ldots, n, n, \ldots, 1.$$

In addition

$$\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_m = -\varepsilon_{m+1} = \ldots = -\varepsilon_{m+n} = \varepsilon_{n+m+1} = \ldots = \varepsilon_{2n+2m} = 1.$$

The inhomogeneous terms $\tilde{c}_A$ are given by

$$\tilde{c}_1 = \hbar, \tilde{c}_2 = \hbar r, \ldots, \tilde{c}_m = \hbar r^{m-1}$$

$$r^2 \tilde{c}_{m+1} = \mathrm{i}\hbar r^{1-m} + \mathrm{i}\hbar (r^2-1)\left[\frac{r^m - r^{-m}}{r - r^{-1}}\right]$$

$$\tilde{c}_{m+2} = \tilde{c}_{m+1}/r, \ldots, \tilde{c}_{m+n} = \tilde{c}_{m+n-1}/r.$$

These $O(\hbar)$ constant terms lead to the following orthosymplectic constraint (summation over A from 1 through $2n+2m$ is understood)

$$\begin{aligned} i^{\pi(A)}\varepsilon_A r^{\hat{A}} x^{A'} x^A &= \mathrm{i} r^{n-(m-1)}\tilde{c}_1 + \mathrm{i} r^{n-(m-2)}\tilde{c}_2 + \ldots + \mathrm{i} r^n \tilde{c}_m \\ &\quad - r^n \tilde{c}_{m+1} - r^{n-1}\tilde{c}_{m+2} + \ldots - r^{-1}\tilde{c}_{m+n} \\ &= \mathrm{i}\hbar r^n [m] - \mathrm{i}\hbar [n]\{r^{-m} + (r - r^{-1})[m]\}. \end{aligned} \tag{11}$$

Equation (11) is evidently a generalization of (4). The special cases $m=0$ and $n=0$ of this orthosymplectic constraint correspond respectively to symplectic and orthogonal constraints for the central length element. The case $m=n$ is also interesting because then the above constraint vanishes identically.

On ignoring the $O(\hbar)$ constant terms, the quadratic algebra $x \otimes x$ is expressible explicitly in terms of the R-matrix for $OSp(2m/2n)$. We have

$$x^B x^A = \frac{1}{r} R^{AB}{}_{CD} x^C x^D \tag{12}$$

with

$$R^{AB}{}_{CD} = \frac{\mathrm{sign}(A,B)}{f_{AB}} r \delta^A{}_C \delta^B{}_D + (r - r^{-1})\delta^A{}_D \delta^B{}_C \Theta^{AB} + (r-r^{-1}) i^{\pi(A)-\pi(C)} \mathrm{sign}(A,A) \mathscr{A}^A{}_C \Theta^A{}_C \delta^{AB'} \delta_{CD'} \tag{13}$$

where

$$\mathscr{A}^A{}_B = \varepsilon_{A'}\varepsilon_B r^{\hat{A}-\hat{B}}$$

and εs and hatted numbers are defined as above. Equation (13) is the R-matrix which is a multiparameter solution of the quantum Yang-Baxter equation for $OSp(2m/2n)$.

Let us give the simplest non-trivial example of the general considerations given above.

OSp(2/2). In this case there are two independent parameters. The complete set of commutation relations among the phase space variables are (here $pq = r^2$):

$$
\begin{aligned}
&x^1x^2 = px^2x^1 \qquad x^1x^3 = qx^3x^1 \\
&x^2x^4 = px^4x^2 \qquad x^3x^4 = qx^4x^3 \\
&(x!)^2 = (x^4)^2 = 0 \\
&x^4x^1 = \hbar - x^1x^4 \\
&x^3x^2 = \mathrm{i}\hbar + r^{-2}x^2x^3 - \mathrm{i}(r^2-1)r^{-2}x^1x^4.
\end{aligned}
\tag{14}
$$

The orthosymplectic constraint reads as

$$\mathrm{i}rx^{1'}x^1 - rx^{2'}x^2 + r^{-1}x^{3'}x^3 + \mathrm{i}r^{-1}x^{4'}x^4 = 0. \tag{15}$$

The non-vanishing elements of the R-matrix for OSp(2/2) are:

$$
\begin{aligned}
&R^{11}_{11} = R^{44}_{44} = -r^{-1} \qquad R^{22}_{22} = R^{33}_{33} = r \\
&R^{24}_{24} = R^{31}_{31} = R^{12}_{12} = R^{43}_{43} = r^{-1}q \qquad R^{21}_{21} = R^{34}_{34} = R^{13}_{13} = R^{42}_{42} = rq^{-1} \\
&R^{42}_{24} = R^{31}_{13} = R^{21}_{12} = R^{43}_{34} = r - r^{-1} \qquad R^{41}_{14} = (r - r^{-1})(1 - r^{-2}) \\
&R^{32}_{23} = (r - r^{-1})(1 + r^{-2}) \qquad R^{23}_{23} = r^{-1} \\
&R^{32}_{32} = r^{-1} \qquad R^{14}_{14} = -r^2 \qquad R^{41}_{41} = -r \\
&R^{32}_{14} = R^{41}_{23} = +\mathrm{i}r^{-2}(r - r^{-1}) \qquad R^{41}_{32} = R^{23}_{14} = -\mathrm{i}(r - r^{-1}).
\end{aligned}
\tag{16}
$$

We would like to conclude by making the following pertinent remarks:

(1) Equation (9) has not been asserted by us in this work; it follows as a consequence of the definitions of the supercoordinates x^A and the algebra given in table 2. The first line of (9) need not cause any confusion due to f_{AB} occurring on both sides because this is merely an alternate way of expressing f_{AA} and f_{BA} $(B > A)$ in terms of our independent parameters r^2 and f_{AB}. Explicitly $f_{AA} = (r^2)^{\pi(A)}$ and $f_{BA} = r^2/f_{AB}$.

(2) To further clarify the meaning of hatted supernumerary indices it may be added that in the two special cases of $m = 0$ and $n = 0$, respectively, our expressions reduce to:

$$
\begin{aligned}
&\hat{1}, \hat{2}, \ldots, \hat{n} = n, n-1, \ldots, 1 \\
&\hat{1}, \hat{2}, \ldots, \hat{m} = -(m-1), -(m-2), \ldots, 0.
\end{aligned}
$$

(3) In this work where we are interested in deformation of phase space, we have assumed that each of the $2n+2m$ variables defining the super phase space is real. This implies that all the deformation parameters are pure phases. However there exists another reality structure. The calculus given in table 1 (2) is automorphic under

$$\overline{z^i} = \partial_i \qquad \overline{\partial_i} = z^i \qquad (\overline{z^I} = \partial_I, \bar{\partial}_I = z^I)$$

provided $|q| = r$ and r is real. If we impose this automorphism, we will be led to the oscillator algebra given by Pusz and Woronowicz.

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